

ON EULER'S FORMULAE FOR DOUBLE ZETA VALUES

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ABSTRACT. In 1776, L. Euler proposed three methods, called *prima methodus*, *secunda methodus* and *tertia methodus*, to calculate formulae for double zeta values. However strictly speaking, his last two methods are mathematically incomplete and require more precise reformulation and more sophisticated arguments for their justification. In this paper, we reformulate his formulae, give their rigorous proofs and also clarify that the formulae can be derived from the extended double shuffle relations.

1. EULER'S METHODS

In 1776, L. Euler published a celebrated paper *Meditations circa singulare serierum genus*¹ written in Latin which means 'Meditations about a singular type of series' in English. It is said to be the first publication in history where multiple zeta values (actually only double zeta values) were introduced. In the paper he proposed three methods to calculate certain relations among double zeta values, which he called *prima methodus*, *secunda methodus* and *tertia methodus*. Here we explain his methods with his idea and point out the steps that would be considered insufficient.

1.1. Prima methodus. In the paper, he studied the series $1 + \frac{1}{2^m} \left(1 + \frac{1}{2^n}\right) + \frac{1}{3^m} \left(1 + \frac{1}{2^n} + \frac{1}{3^n}\right) + \frac{1}{4^m} \left(1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n}\right) + \cdots$ which he denoted by the unconventional notation $\int \frac{1}{z^m} \left(\frac{1}{y^n}\right)$ and also the series $1 + \frac{1}{2^m} + \frac{1}{3^m} + \frac{1}{4^m} + \cdots$ which he again denoted by the notation $\int \frac{1}{z^m}$. In modern language, they are nothing but the *double zeta star value*

$$\zeta^*(n, m) := \sum_{0 < k_1 \leq k_2} \frac{1}{k_1^n k_2^m} = \zeta(n, m) + \zeta(m + n)$$

for $n \in \mathbb{Z}_{>0}$ and $m \in \mathbb{Z}_{>1}$, and the *Riemann zeta value* $\zeta(m) := \sum_{k=1}^{\infty} \frac{1}{k^m}$ for $m \in \mathbb{Z}_{>1}$ respectively. Here we recall the *double zeta value*

$$\zeta(n, m) := \sum_{0 < k_1 < k_2} \frac{1}{k_1^n k_2^m}$$

for $n \in \mathbb{Z}_{>0}$ and $m \in \mathbb{Z}_{>1}$. By multiplying these series, he obtained the **formula of prima methodus** (cf. [1] p.144)

$$\int \frac{1}{z^m} \left(\frac{1}{y^n}\right) + \int \frac{1}{z^n} \left(\frac{1}{y^m}\right) = \int \frac{1}{z^m} \cdot \int \frac{1}{z^n} + \int \frac{1}{z^{m+n}}.$$

In modern language, this says

$$(1) \quad \zeta^*(n, m) + \zeta^*(m, n) = \zeta(n)\zeta(m) + \zeta(m + n).$$

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¹Downloadable from <http://eulerarchive.maa.org>.

It is nothing but the *harmonic product formula*,

$$(2) \quad \zeta(m, n) + \zeta(n, m) + \zeta(m + n) = \zeta(m)\zeta(n),$$

which is known to hold for $m, n \in \mathbb{Z}_{>1}$.

1.2. Secunda methodus. This subsection summarizes [1] pp.144–149.

Firstly, Euler began with a partial fraction decomposition (cf. [1] pp.145–146)

$$\begin{aligned} \frac{1}{x^n(x+a)^m} &= \frac{1}{a^n} \cdot \frac{1}{x^m} - \frac{n}{1 \cdot a^{n+1}} \cdot \frac{1}{x^{m-1}} + \frac{n(n+1)}{1 \cdot 2a^{n+2}} \cdot \frac{1}{x^{m-2}} \\ &\quad - \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3a^{n+3}} \cdot \frac{1}{x^{m-3}} + \text{etc.} \\ &\quad \pm \frac{1}{a^m} \cdot \frac{1}{(x+a)^n} \pm \frac{m}{1 \cdot a^{m+1}} \cdot \frac{1}{(x+a)^{n-1}} \pm \frac{m(m+1)}{1 \cdot 2a^{m+2}} \cdot \frac{1}{(x+a)^{n-2}} \\ &\quad \pm \frac{m(m+1)(m+2)}{1 \cdot 2 \cdot 3a^{m+3}} \cdot \frac{1}{(x+a)^{n-3}} \pm \text{etc.} \end{aligned}$$

In modern language it reads

$$(3) \quad \frac{1}{x^n(x+a)^m} = \sum_{i=0}^{m-1} (-1)^i \binom{n+i-1}{i} \frac{1}{a^{n+i}} \frac{1}{x^{m-i}} + (-1)^m \sum_{j=0}^{n-1} \binom{m+j-1}{j} \frac{1}{a^{m+j}} \frac{1}{(x+a)^{n-j}}.$$

Secondly, he put² $s_a := \sum_{x=1}^{\infty} \frac{1}{x^n(x+a)^m}$, and calculated as follows (cf. [1] pp.146–147):

$$\begin{aligned} s_a &= \frac{1}{a^n} \int \frac{1}{z^m} - \frac{n}{1 \cdot a^{n+1}} \int \frac{1}{z^{m-1}} + \frac{n(n+1)}{1 \cdot 2a^{n+2}} \int \frac{1}{z^{m-2}} - \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3a^{n+3}} \int \frac{1}{z^{m-3}} + \text{etc.} \\ &\quad \pm \frac{1}{a^m} \int \frac{1}{z^n} \pm \frac{m}{1 \cdot a^{m+1}} \int \frac{1}{z^{n-1}} \pm \frac{m(m+1)}{1 \cdot 2a^{m+2}} \int \frac{1}{z^{n-2}} \pm \frac{m(m+1)(m+2)}{1 \cdot 2 \cdot 3a^{m+3}} \int \frac{1}{z^{n-3}} \pm \text{etc.} \\ &\quad \mp \frac{1}{a^m} \left(1 + \frac{1}{2^n} + \frac{1}{3^n} + \cdots + \frac{1}{a^n} \right) \mp \frac{m}{1 \cdot a^{m+1}} \left(1 + \frac{1}{2^{n-1}} + \frac{1}{3^{n-1}} + \cdots + \frac{1}{a^{n-1}} \right) \\ &\quad \mp \frac{m(m+1)}{1 \cdot 2a^{m+2}} \left(1 + \frac{1}{2^{n-2}} + \frac{1}{3^{n-2}} + \cdots + \frac{1}{a^{n-2}} \right) \mp \frac{m(m+1)(m+2)}{1 \cdot 2 \cdot 3a^{m+3}} \left(1 + \frac{1}{2^{n-3}} + \frac{1}{3^{n-3}} + \cdots + \frac{1}{a^{n-3}} \right) \\ &\quad \mp \text{etc.} \end{aligned}$$

In modern language, this is written as

$$\begin{aligned} s_a &= \sum_{x=1}^{\infty} \frac{1}{x^n(x+a)^m} = \sum_{i=0}^{m-1} \sum_{x=1}^{\infty} (-1)^i \binom{n+i-1}{i} \frac{1}{a^{n+i}} \frac{1}{x^{m-i}} \\ &\quad + (-1)^m \sum_{j=0}^{n-1} \sum_{x=1}^{\infty} \binom{m+j-1}{j} \frac{1}{a^{m+j}} \frac{1}{(x+a)^{n-j}}. \end{aligned}$$

²We note that in [1], it is simply denoted by s , which could give rise to confusion in this text.

Remark 1. By using (3), he came to the above equation in the following way:

$$\begin{aligned}
& \sum_{x=1}^{\infty} \frac{1}{x^n (x+a)^m} \\
&= \sum_{x=1}^{\infty} \left\{ \sum_{i=0}^{m-1} (-1)^i \binom{n+i-1}{i} \frac{1}{a^{n+i}} \frac{1}{x^{m-i}} + (-1)^m \sum_{j=0}^{n-1} \binom{m+j-1}{j} \frac{1}{a^{m+j}} \frac{1}{(x+a)^{n-j}} \right\} \\
&\stackrel{!}{=} \sum_{i=0}^{m-1} \sum_{x=1}^{\infty} (-1)^i \binom{n+i-1}{i} \frac{1}{a^{n+i}} \frac{1}{x^{m-i}} + (-1)^m \sum_{j=0}^{n-1} \sum_{x=1}^{\infty} \binom{m+j-1}{j} \frac{1}{a^{m+j}} \frac{1}{(x+a)^{n-j}}.
\end{aligned}$$

Here we alert reader to the fact that validity of the above equation $\stackrel{!}{=}$ is really problematic. We can not exchange two summations because the right hand side of this equation does not converge absolutely.

Thirdly, he considered $\sum_{a=1}^{\infty} s_a$ and calculated as follows (cf. [1] pp.147–148):

$$\begin{aligned}
\sum_{a=1}^{\infty} s_a &= \int \frac{1}{z^n} \cdot \int \frac{1}{z^m} - \frac{n}{1} \int \frac{1}{z^{n+1}} \cdot \int \frac{1}{z^{m-1}} + \frac{n(n+1)}{1 \cdot 2} \int \frac{1}{z^{n+2}} \cdot \int \frac{1}{z^{m-2}} \\
&\quad - \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3} \int \frac{1}{z^{n+3}} \cdot \int \frac{1}{z^{m-3}} + \text{etc.} \\
&\pm \int \frac{1}{z^m} \cdot \int \frac{1}{z^n} \mp \int \frac{1}{z^m} \left(\frac{1}{y^n} \right) \pm \frac{m}{1} \int \frac{1}{z^{m+1}} \cdot \int \frac{1}{z^{n-1}} \mp \frac{m}{1} \int \frac{1}{z^{m+1}} \left(\frac{1}{y^{n-1}} \right) \\
&\pm \frac{m(m+1)}{1 \cdot 2} \int \frac{1}{z^{m+2}} \cdot \int \frac{1}{z^{n-2}} \mp \frac{m(m+1)}{1 \cdot 2} \int \frac{1}{z^{m+2}} \left(\frac{1}{y^{n-2}} \right) \\
&\pm \frac{m(m+1)(m+2)}{1 \cdot 2 \cdot 3} \int \frac{1}{z^{m+3}} \cdot \int \frac{1}{z^{n-3}} \mp \frac{m(m+1)(m+2)}{1 \cdot 2 \cdot 3} \int \frac{1}{z^{m+3}} \left(\frac{1}{y^{n-3}} \right) \\
&\pm \text{etc.}
\end{aligned}$$

In modern language, it means

$$\begin{aligned}
(4) \quad \zeta(m, n) &= \sum_{a=1}^{\infty} s_a = \sum_{a=1}^{\infty} \sum_{i=0}^{m-1} (-1)^i \binom{n+i-1}{i} \frac{1}{a^{n+i}} \zeta(m-i) \\
&\quad + (-1)^m \sum_{a=1}^{\infty} \sum_{j=0}^{n-1} \binom{m+j-1}{j} \frac{1}{a^{m+j}} \left\{ \zeta(n-j) - \sum_{k=1}^a \frac{1}{k^{n-j}} \right\} \\
&= \sum_{i=0}^{m-1} (-1)^i \binom{n+i-1}{i} \zeta(n+i) \zeta(m-i) \\
&\quad + (-1)^m \sum_{j=0}^{n-1} \binom{m+j-1}{j} \left\{ \zeta(m+j) \zeta(n-j) - \zeta^*(n-j, m+j) \right\}.
\end{aligned}$$

The above is a **key formula** of secunda methodus and tertia methodus. Finally he substituted the formula (4) into (2) and obtained the **formula of secunda**

methodus (cf. [1] pp.148–149).

$$\begin{aligned}
& \int \frac{1}{z^m} \cdot \int \frac{1}{z^n} - \int \frac{1}{z^{m+n}} \\
&= (1 \pm 1) \int \frac{1}{z^m} \cdot \int \frac{1}{z^n} \mp \int \frac{1}{z^m} \left(\frac{1}{y^n} \right) \\
&\quad - \frac{m}{1} (1 \mp 1) \int \frac{1}{z^{m+1}} \cdot \int \frac{1}{z^{n-1}} \mp \frac{m}{1} \int \frac{1}{z^{m+1}} \left(\frac{1}{y^{n-1}} \right) \\
&\quad + \frac{m(m+1)}{1 \cdot 2} (1 \pm 1) \int \frac{1}{z^{m+2}} \cdot \int \frac{1}{z^{n-2}} \mp \frac{m(m+1)}{1 \cdot 2} \int \frac{1}{z^{m+2}} \left(\frac{1}{y^{n-2}} \right) \\
&\quad - \frac{m(m+1)(m+2)}{1 \cdot 2 \cdot 3} (1 \mp 1) \int \frac{1}{z^{m+3}} \cdot \int \frac{1}{z^{n-3}} \mp \frac{m(m+1)(m+2)}{1 \cdot 2 \cdot 3} \int \frac{1}{z^{m+3}} \left(\frac{1}{y^{n-3}} \right) \\
&\quad \pm \text{etc.} \\
&\quad + (1 \pm 1) \int \frac{1}{z^n} \cdot \int \frac{1}{z^m} \mp \int \frac{1}{z^n} \left(\frac{1}{y^m} \right) \\
&\quad - \frac{n}{1} (1 \mp 1) \int \frac{1}{z^{n+1}} \cdot \int \frac{1}{z^{m-1}} \mp \frac{n}{1} \int \frac{1}{z^{n+1}} \left(\frac{1}{y^{m-1}} \right) \\
&\quad + \frac{n(n+1)}{1 \cdot 2} (1 \pm 1) \int \frac{1}{z^{n+2}} \cdot \int \frac{1}{z^{m-2}} \mp \frac{n(n+1)}{1 \cdot 2} \int \frac{1}{z^{n+2}} \left(\frac{1}{y^{m-2}} \right) \\
&\quad - \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3} (1 \mp 1) \int \frac{1}{z^{n+3}} \cdot \int \frac{1}{z^{m-3}} \mp \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3} \int \frac{1}{z^{n+3}} \left(\frac{1}{y^{m-3}} \right) \pm \text{etc.}
\end{aligned}$$

In modern language, it is translated into the following:

(5)

$$\begin{aligned}
\zeta(m)\zeta(n) - \zeta(m+n) &= \sum_{i=0}^{m-1} (-1)^i \binom{n+i-1}{i} \zeta(n+i) \zeta(m-i) \\
&\quad + (-1)^m \sum_{j=0}^{n-1} \binom{m+j-1}{j} \left\{ \zeta(m+j) \zeta(n-j) - \zeta^*(n-j, m+j) \right\} \\
&\quad + \sum_{i=0}^{n-1} (-1)^i \binom{m+i-1}{i} \zeta(m+i) \zeta(n-i) \\
&\quad + (-1)^n \sum_{j=0}^{m-1} \binom{n+j-1}{j} \left\{ \zeta(n+j) \zeta(m-j) - \zeta^*(m-j, n+j) \right\}.
\end{aligned}$$

1.3. Tertia methodus. Euler proposed another method in [1] pp.168–170. He introduced the following unconventional notations in [1] pp.165–166: $p^\mu = p^\nu := \zeta(\mu)\zeta(\nu)$, $p^\lambda := \zeta(\lambda)$, $q^\mu := \zeta^*(\nu, \mu) = \zeta^*(\lambda - \mu, \mu)$ with $\mu + \nu = \lambda$ and $\mu, \nu \in \mathbb{Z}_{>0}$. By using these symbols, he rewrote the equations (1) and (4) respectively as follows (cf. [1] pp.169–170):

$$\begin{aligned}
q^m + q^n &= p^m + p^{m+n} = p^n + p^{m+n}. \\
q^n - p^{m+n} &= p^m - \frac{n}{1} p^{m-1} + \frac{n(n+1)}{1 \cdot 2} p^{m-2} - \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3} p^{m-3} + \text{etc.} \\
&\quad \pm q^n \pm \frac{m}{1} q^{n-1} \pm \frac{m(m+1)}{1 \cdot 2} q^{n-2} \pm \frac{m(m+1)(m+2)}{1 \cdot 2 \cdot 3} q^{n-3} \pm \text{etc.} \\
&\quad \mp p^{m+n} \mp \frac{m}{1} p^{m+n} \mp \frac{m(m+1)}{1 \cdot 2} p^{m+n} \mp \frac{m(m+1)(m+2)}{1 \cdot 2 \cdot 3} p^{m+n} \mp \text{etc.}
\end{aligned}$$

By substituting the former equation $p^\mu = q^\mu + q^{m+n-\mu} - p^{m+n}$ into the latter equation, he obtained the following (cf. [1] p.170)

$$\begin{aligned} 0 = & q^m - \frac{n}{1}(q^{m-1} + q^{n+1}) + \frac{n(n+1)}{1 \cdot 2}(q^{m-2} + q^{n+2}) \\ & - \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}(q^{m-3} + q^{n+3}) + \text{etc.} \\ & + \frac{n}{1}p^{m+n} - \frac{n(n+1)}{1 \cdot 2}p^{m+n} + \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}p^{m+n} - \text{etc.} \\ & \pm q^n \pm \frac{m}{1}q^{n-1} \pm \frac{m(m+1)}{1 \cdot 2}q^{n-2} \pm \frac{m(m+1)(m+2)}{1 \cdot 2 \cdot 3}q^{n-3} \pm \text{etc.} \\ & \mp p^{m+n} \mp \frac{m}{1}p^{m+n} \mp \frac{m(m+1)}{1 \cdot 2}p^{m+n} \mp \frac{m(m+1)(m+2)}{1 \cdot 2 \cdot 3}p^{m+n} \mp \text{etc.} \end{aligned}$$

In modern language, this means that, by (1), he transformed (4) into the following:

$$\begin{aligned} (6) \quad \zeta(m, n) = & \sum_{i=0}^{m-1} (-1)^i \binom{n+i-1}{i} \left\{ \zeta(n+i, m-i) + \zeta(m-i, n+i) + \zeta(m+n) \right\} \\ & + (-1)^m \sum_{j=0}^{n-1} \binom{m+j-1}{j} \zeta(m+j, n-j). \end{aligned}$$

This is the **formula of tertia methodus**.

Remark 2. We warn that formulae (4), (5) and (6) contain meaningless values $\zeta(1)$ and $\zeta(m+n-1, 1)$. However, they will be correctly reformulated in Theorem 6 (14), Theorem 7 (17) and (18).

Remark 3. The sum formula of Granville [3] and Zagier (unpublished) in the case of double zeta values is recovered as a special case of (6) for $m = 1$.

2. MAIN RESULT

In this section we give a rigorous reformulation of Euler's problematic formulae and its complete proof in Theorem 6 and 7 by using the generating functions of double zeta values introduced by Gangl, Kaneko, Zagier in [2].

Proposition 4. *Double zeta values enjoy the **double shuffle relations**. Namely, the **shuffle product formula***

$$(7) \quad \zeta(m)\zeta(n) = \sum_{k=0}^{n-1} \binom{m+k-1}{k} \zeta(n-k, m+k) + \sum_{l=0}^{m-1} \binom{n+l-1}{l} \zeta(m-l, n+l),$$

*and the **harmonic product formula***

$$(8) \quad \zeta(m)\zeta(n) = \zeta(m, n) + \zeta(n, m) + \zeta(m+n)$$

hold for $m, n \in \mathbb{Z}_{>1}$.

We recall two notions of regularization of double zeta values along the line of [4]³. Let $m, n \in \mathbb{Z}_{>0}$. It is shown that when $N \rightarrow \infty$, we have

$$\sum_{0 < k < N} \frac{1}{k^m} \sim a_0 + a_1(\log N + \gamma), \quad \sum_{0 < k_1 < k_2 < N} \frac{1}{k_1^m k_2^n} \sim b_0 + b_1(\log N + \gamma) + b_2(\log N + \gamma)^2,$$

³Another reformulation was also given in [5].

with some $a_i, b_i \in \mathbb{R}$ and the Euler's constant $\gamma := \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right)$. Here $f(x) \sim g(x)$ ($x \rightarrow \alpha$) means $\frac{f(x)}{g(x)} \rightarrow 1$ ($x \rightarrow \alpha$) (this α can be infinity). The **harmonic regularized values** $\zeta_*(m)$ and $\zeta_*(m, n)$ are defined as follows:

$$\zeta_*(m) := a_0 + a_1 \cdot T, \quad \zeta_*(m, n) := b_0 + b_1 \cdot T + b_2 \cdot T^2 \in \mathbb{R}[T].$$

In contrast, for $m, n \in \mathbb{Z}_{>0}$, it is also shown that when $\epsilon \rightarrow 0$, we have

$$\sum_{0 < k} \frac{(1 - \epsilon)^k}{k^m} \sim c_0 + c_1(-\log \epsilon), \quad \sum_{0 < k_1 < k_2} \frac{(1 - \epsilon)^{k_2}}{k_1^m k_2^n} \sim d_0 + d_1(-\log \epsilon) + d_2(-\log \epsilon)^2,$$

with some $c_i, d_i \in \mathbb{R}$. For $m, n \in \mathbb{Z}_{>0}$, the **shuffle regularized values** $\zeta_{\text{sh}}(m)$ and $\zeta_{\text{sh}}(m, n)$ are defined as follows:

$$\zeta_{\text{sh}}(m) := c_0 + c_1 \cdot T, \quad \zeta_{\text{sh}}(m, n) := d_0 + d_1 \cdot T + d_2 \cdot T^2 \in \mathbb{R}[T].$$

We remark that

$$(9) \quad \zeta_*(m) = \begin{cases} \zeta_{\text{sh}}(m) = T & \text{if } m = 1, \\ \zeta_{\text{sh}}(m) = \zeta(m) & \text{if } m > 1, \end{cases}$$

and $\zeta_*(m, n) = \zeta_{\text{sh}}(m, n) = \zeta(m, n)$ if $n > 1$.

Proposition 5. *The extended double shuffle relations hold for the regularized values. Namely, for $m, n \in \mathbb{Z}_{>0}$, we have*

(10)

$$\zeta_{\text{sh}}(m)\zeta_{\text{sh}}(n) = \sum_{i=0}^{m-1} \binom{n+i-1}{i} \zeta_{\text{sh}}(m-i, n+i) + \sum_{j=0}^{n-1} \binom{m+j-1}{j} \zeta_{\text{sh}}(n-j, m+j),$$

(11)

$$\zeta_*(m)\zeta_*(n) = \zeta_*(m, n) + \zeta_*(n, m) + \zeta_*(m+n),$$

(12)

$$\begin{aligned} & \sum_{i=0}^{m-1} \binom{n+i-1}{i} \zeta_{\text{sh}}(m-i, n+i) + \sum_{j=0}^{n-1} \binom{m+j-1}{j} \zeta_{\text{sh}}(n-j, m+j) \\ &= \zeta_*(m, n) + \zeta_*(n, m) + \zeta_*(m+n). \end{aligned}$$

We recall generating functions of double zeta values and Riemann zeta values which were introduced in [2]. For $k \in \mathbb{Z}_{>1}$, we put

$$D_k(X, Y) := \sum_{i=1}^{k-1} \zeta_{\text{sh}}(k-i, i) X^{i-1} Y^{k-i-1}, \quad Q_k(X, Y) := \sum_{i=1}^{k-1} \zeta_{\text{sh}}(i) \zeta_{\text{sh}}(k-i) X^{i-1} Y^{k-i-1}.$$

They showed in [2] p.80 (25) the following relation

(13)

$$D_k(X+Y, Y) + D_k(X+Y, X) = Q_k(X, Y),$$

which is a reformulation of shuffle product formula (10).

The following is a reformulation of Euler's problematic key formula (4).

Theorem 6. *For $m, n \in \mathbb{Z}_{>1}$,*

(14)

$$\zeta(m, n) = P(m, n),$$

where

$$(15) \quad P(m, n) := \sum_{i=0}^{m-1} (-1)^i \binom{n+i-1}{i} \zeta(n+i) \zeta_{\text{III}}(m-i) \\ + (-1)^m \sum_{j=0}^{n-1} \binom{m+j-1}{j} \left\{ \zeta(m+j) \zeta_{\text{III}}(n-j) - \zeta^*(n-j, m+j) \right\}.$$

Proof. By changing a variable X to $X - Y$ and putting $k = m + n$ in (13), we get

$$(16) \quad D_{m+n}(X, Y) + D_{m+n}(X, X - Y) = Q_{m+n}(X - Y, Y).$$

We note that

$$\sum_{i=0}^{m-1} (-1)^i \binom{n+i-1}{i} \zeta(n+i) \zeta_{\text{III}}(m-i)$$

is the coefficient of $X^{n-1}Y^{m-1}$ in $Q_{m+n}(X - Y, Y)$, while

$$(-1)^m \sum_{j=0}^{n-1} \binom{m+j-1}{j} \left\{ \zeta(m+j) \zeta_{\text{III}}(n-j) - \zeta^*(n-j, m+j) \right\}$$

is the coefficient of $X^{n-1}Y^{m-1}$ in $-D_{m+n}(X, X - Y)$ and $\zeta(m, n)$ is the coefficient of $X^{n-1}Y^{m-1}$ in $D_{m+n}(X, Y)$. Therefore it follows from (16) that $P(m, n) = \zeta(m, n)$. \square

The following theorem gives a reformulation of Euler's secunda methodus (5) and tertia methodus (6).

Theorem 7. For $m, n \in \mathbb{Z}_{>1}$, we have

$$(17) \quad \zeta(m)\zeta(n) - \zeta(m+n) = P(m, n) + P(n, m),$$

and

$$(18) \quad \zeta(m, n) = \sum_{i=0}^{m-1} (-1)^i \binom{n+i-1}{i} \left\{ \zeta_*(n+i, m-i) + \zeta(m-i, n+i) + \zeta(m+n) \right\} \\ + (-1)^m \sum_{j=0}^{n-1} \binom{m+j-1}{j} \zeta_*(m+j, n-j),$$

Proof.

Secunda methodus: By Theorem 6, $P(m, n) = \zeta(m, n)$ for $m, n \in \mathbb{Z}_{>1}$. So by using the harmonic product formula (8), we obtain

$$P(m, n) + P(n, m) = \zeta(m, n) + \zeta(n, m) = \zeta(m)\zeta(n) - \zeta(m+n).$$

Hence (17) is shown.

Tertia methodus: Again by Theorem 6, the extended double shuffle relations, (10), (11) and (12), yield

$$\zeta(m, n) = P(m, n) \\ = \sum_{i=0}^{m-1} (-1)^i \binom{n+i-1}{i} \zeta(n+i) \zeta_{\text{III}}(m-i) + (-1)^m \sum_{j=0}^{n-1} \binom{m+j-1}{j} \zeta_{\text{III}}(m+j, n-j), \\ = \sum_{i=0}^{m-1} (-1)^i \binom{n+i-1}{i} \zeta(n+i) \zeta_{\text{III}}(m-i) + (-1)^m \sum_{j=0}^{n-1} \binom{m+j-1}{j} \left\{ \zeta(m+n-1) T \right. \\ \left. - \zeta(1, n+m-1) - \sum_{k=1}^{m+n-2} \zeta(m+n-1-k, 1+k) \right\},$$

by using the equation (12),

$$\begin{aligned}
&= \sum_{i=0}^{m-1} (-1)^i \binom{n+i-1}{i} \zeta(n+i) \zeta_{\text{III}}(m-i) + (-1)^m \sum_{j=0}^{n-1} \binom{m+j-1}{j} \left\{ \zeta(m+n-1) T \right. \\
&\quad \left. - \zeta(1, n+m-1) - \zeta(m+n) - \zeta_{\text{III}}(1, n+m-1) \right. \\
&\quad \left. + \sum_{j=0}^0 \binom{m+n-2+j}{j} \zeta_{\text{III}}(1-j, m+n-1+j) \right\} \\
&= \sum_{i=0}^{m-1} (-1)^i \binom{n+i-1}{i} \zeta(n+i) \zeta_{\text{III}}(m-i) + (-1)^m \sum_{j=0}^{n-1} \binom{m+j-1}{j} \zeta_*(m+j, n-j),
\end{aligned}$$

by Remark 9,

$$= \sum_{i=0}^{m-1} (-1)^i \binom{n+i-1}{i} \zeta(n+i) \zeta_*(m-i) + (-1)^m \sum_{j=0}^{n-1} \binom{m+j-1}{j} \zeta_*(m+j, n-j)$$

and finally by the harmonic product formula (11),

$$\begin{aligned}
&= \sum_{i=0}^{m-1} (-1)^i \binom{n+i-1}{i} \left\{ \zeta_*(n+i, m-i) + \zeta(m-i, n+i) + \zeta(m+n) \right\} \\
&\quad + (-1)^m \sum_{j=0}^{n-1} \binom{m+j-1}{j} \zeta_*(m+j, n-j).
\end{aligned}$$

Hence (18) is shown. \square

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